

SPECIALTY FUNCTIONS  
FOR  
SINGULARITY MECHANICS PROBLEMS<sup>1</sup>

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With the increasing use of new materials and elevated temperature applications studies involving more accurate predictions on the values of the field variables and more efficient computational methods are receiving considerable attention.

The focus of this research is in the development of more accurate and efficient advanced methods for solution of singular problems encountered in mechanics. At present, finite element methods in conjunction with special functions, boolean sum and blending interpolations are being considered. In dealing with systems which contain a singularity, special finite elements are being formulated to be used in singular regions. Further, special transition elements are being formulated to couple the special element to the mesh that model the rest of the system, and to be used in conjunction with 1-D, 2-D and 3-D elements within the same mesh. Computational simulation with a least squares fit is being utilized to construct special elements, if there is an unknown singularity in the system.

A novel approach is taken in formulation of the elements in that; i) the material properties are modified to include time, temperature, coordinate and stress dependant behavior within the element, ii) material properties vary at nodal points of the elements, iii) a hidden-symbolic computation scheme is developed and utilized in formulating the elements, and iv) special functions and boolean sum are utilized in order to interpolate the field variables and their derivatives along the boundary of the elements.

It may be noted that the proposed methods are also applicable to fluids and coupled problems.

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## O U T L I N E

- MOTIVATION
- OBJECTIVE
- APPROACH
- NUMERICAL CONSIDERATIONS
- CONCLUDING REMARKS

O B J E C T I V E

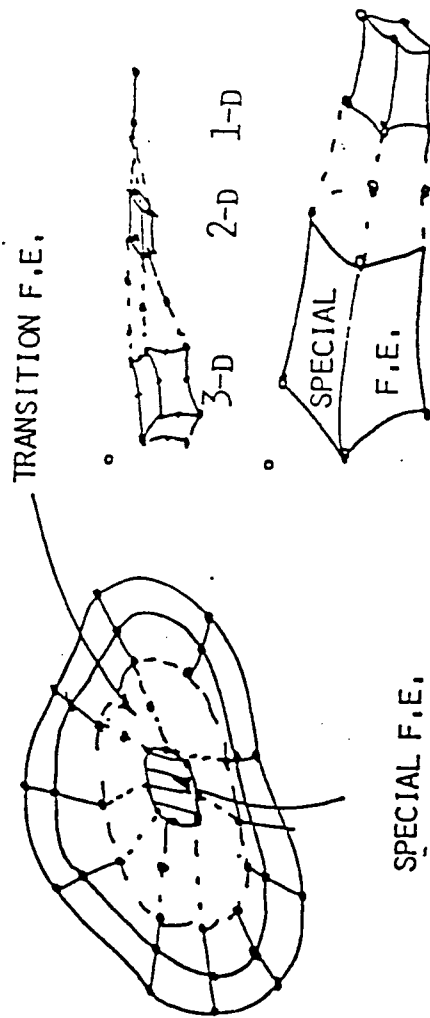
DEVELOPMENT OF ADVANCED METHODS TO SOLVE SINGULARITY PROBLEMS  
ENCOUNTERED IN MECHANICS

# SPECIAL FUNCTIONS — SINGULAR PROBLEMS

KNOWN SINGULARITY

$\delta$  DIRAC

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. .  
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SOLID

GEOMETRY (IRREGULAR)

NONLINEARITIES, MATERIAL  
LOADING

UNKNOWN SINGULARITY

COMPUTATIONAL SIMULATION

FLUID

FLOW SEP.

SHOCK FORMATION

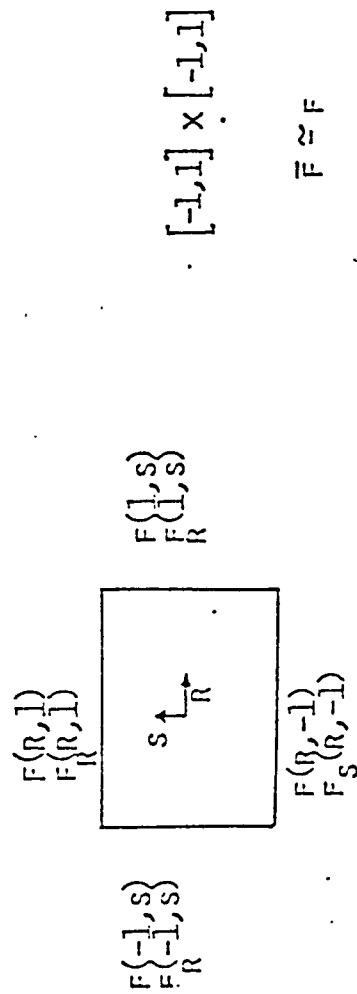
HEAT SOURCE  
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## SPECIAL FUNCTIONS - SINGULAR PROBLEMS

Special finite elements are being formulated for solution of singular problems encountered in mechanics such as due to sudden change in geometry or loading, shock formation, and heat source. In cases where the singularity is known prior to the analysis, known singularity, the analytical solution available is incorporated in the element formulation. If the singularity is an unknown type, then a computational simulation together with a least squares fit is being utilized with special functions to formulate the special elements. In order to blend the special elements to the mesh with traditional finite elements, the transition finite elements are being formulated.

# ELEMENT FORMULATION

- BOOLEAN SUM  $P[F(r,s)] = (P_R \oplus P_S) F$
- BLENDING INTERPOLANTS



FOR R DIRECTION	$P_R[F(r,s)]$	$= F(-1,s) c_1(r) + F(1,s) c_2(r) + F(-1,-1) c_1(s) + F(1,-1) c_2(s)$
S DIRECTION	$P_S[F(r,s)]$	$= F(r,1) c_1(s) + F(r,-1) c_2(s) + F(-1,1) c_1(s) + F(-1,-1) c_2(s)$
	$\bar{F}(r,s) \Big _{R=1} = F(1,s)$	F "INTERPOLATION FUNCTION"
	$\bar{F}(r,s) \Big _{R=-1} = F(-1,s)$	C "BLENDING FUNCTION"
	$\bar{F}(r,s) \Big _{S=1} = F(r,1)$	P "PROJECTOR OPERATOR"
	$\bar{F}(r,s) \Big _{S=-1} = F(r,-1)$	

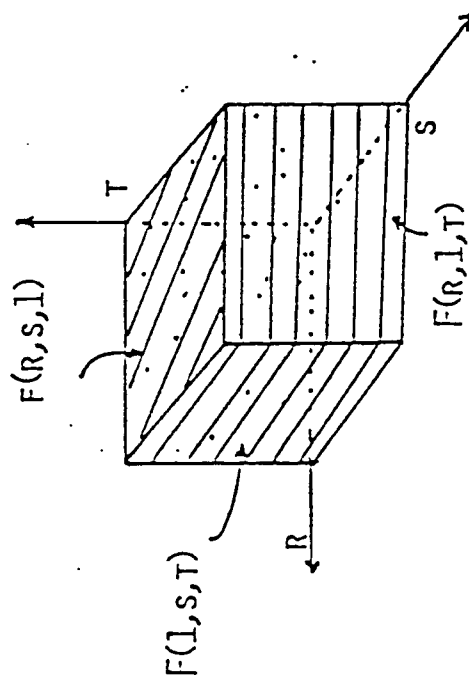
#### ELEMENT FORMULATION

The boolean sum is utilized to construct the finite elements. As it is illustrated in figure the field variable  $F$  is approximated as  $\bar{F}$ . It may be noted that along the boundary of the element the approximation function and its normal derivatives are identical to the field variable and its normal derivatives respectively. The projector operators are given in terms of the interpolation functions and the blending functions.

# FINITE ELEMENT FORMULATION 3-D

$$\omega \equiv [0,1] \times [0,1] \times [0,1]$$

$$\bar{\omega} \equiv \omega \cup \partial\omega$$



THE BOOLEAN SUM

$$P[F(R, S, T)] \equiv P_R \oplus (P_S \oplus P_T) [F]$$

OR

$$P[F] \equiv (P_R + P_S + P_T - P_R P_S - P_S P_T - P_R P_T + P_R P_S P_T)[F]$$

$$\bar{F}(R, S, T) \equiv F(R, S, T).$$

FOR ALL  $(R, S, T)$  ON  $\partial\omega$

SURFACES

$$P_R[F] = RF(L, S, T) + (1-R)F(0, S, T)$$

$$P_S[F] = SF(R, 1, T) + (1-S)F(R, 0, T)$$

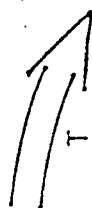
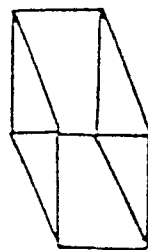
$$P_T[F] = TF(R, S, 1) + (1-T)F(R, S, 0)$$



### FINITE ELEMENT FORMULATION 3-D

The boolean sum is expressed in terms of the three projector operators along R, S and T directions. The continuity requirement for this example is for the field variable itself only. The approximation function exactly matches the values of the field variable on the boundary surface. Therefore, it is anticipated that the results obtained from the analysis will give better approximations on the field variable.

# TRANSFORMATION OF THE DOMAIN



$$T[XYZ] \equiv (P_R \oplus P_S \oplus P_T) [XYZ]$$

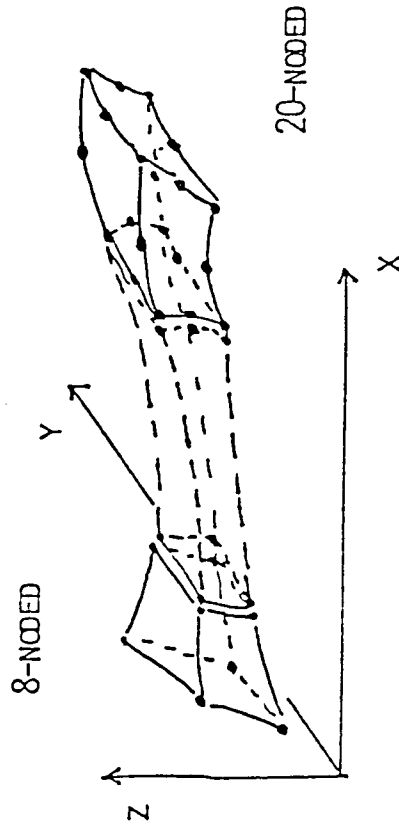
$$XYZ = (X, Y, Z)^T$$

#### TRANSFORMATION OF THE DOMAIN

Transfinite interpolation formula is being utilized in order to account for arbitrary geometry. The unit cube is mapped onto the curved domain by utilizing the projectors along  $r, s$ , and  $t$  directions.

# FORMULATION OF A TRANSITION ELEMENT

TRANSITION F.E.



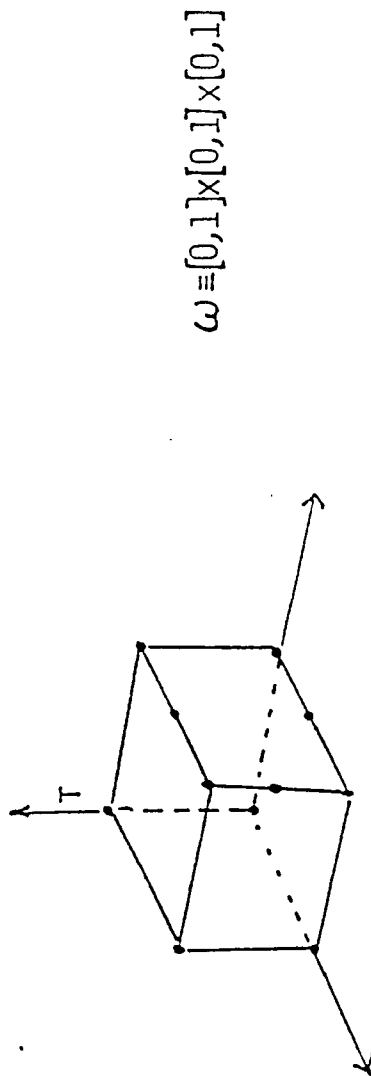
## REQUIREMENTS

- CONNECTS TWO VARIABLY DEGREES OF FREEDOM 3-D ELEMENTS
- ANISOTROPIC MATERIAL  
(ELEVATED TEMPERATURES)
- NONLINEAR MATERIAL BEHAVIOR
- LARGE DEFORMATIONS
- MATERIAL PROPERTIES KNOWN AT THE NODES

#### FORMULATION OF A TRANSITION ELEMENT

In order to connect three dimensional meshes obtained from different traditional finite elements, various 3-D finite elements are being formulated. For the elements formulated anisotropic material behavior and large deformation effects are also being included in the formulation.

# A 12-NODE TRANSITION FINITE ELEMENT



$$P_R[u] \equiv Ru(1,s,t) + (1-R)u(0,s,t)$$

$$P_S[u] \equiv Su(r,1,t) + (1-S)u(r,0,t)$$

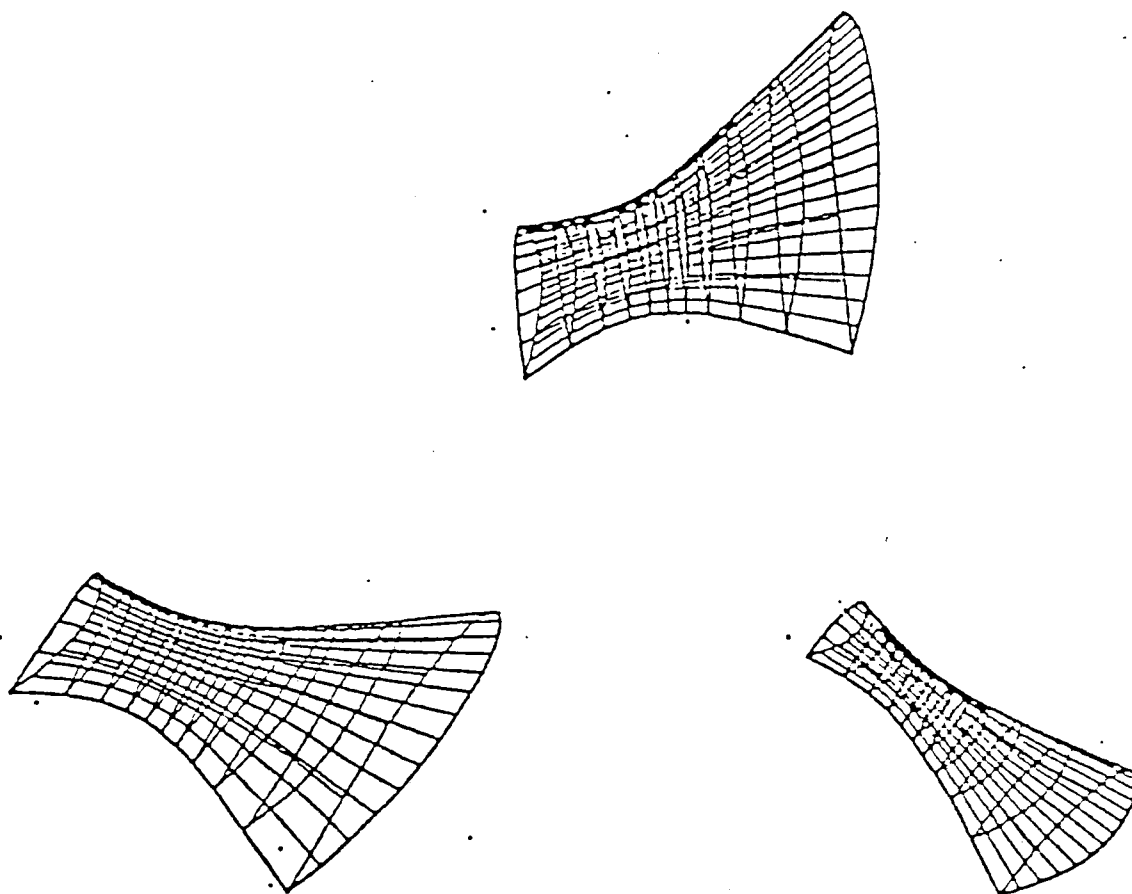
$$P_T[u] \equiv Tu(r,s,1) + (1-T)u(r,s,0)$$

$$\bar{U}(r,s,t) \equiv P_R \oplus (P_S \oplus P_T)[u]$$

$$\bar{U}(r,s,t) \equiv u(r,s,t) \quad \text{ON } \partial\omega$$

#### A 12-NODE TRANSITION ELEMENT

A 12-node three dimensional transition element is formulated and is being tested. The formulation is based on the boolean sum, and the integrations involved in element matrices are performed exactly by utilizing a hidden-symbolic computation approach.





### Modeling of a closed shell

By utilizing similar interpolation and blending functions various size finite element meshes are obtained from a closed shell structure.

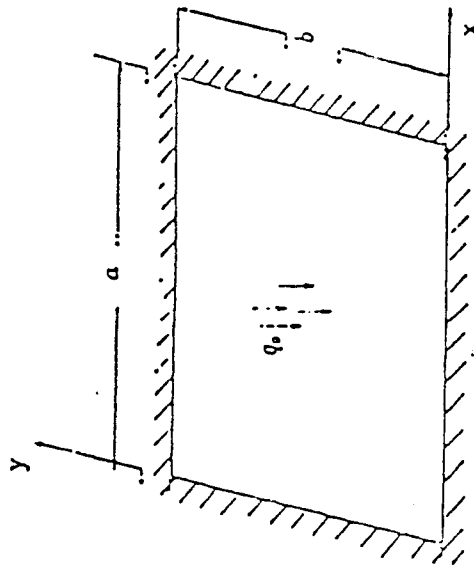


Figure 5.4: Uniformly loaded rectangular plate with built-in edges.

For this example the results are compared with the solutions given by reference[1]. The comparison is made in the form of plot of the plate center deflection  $w$ , versus the number of elements used in the discretization. The results are tabulated in table 5.3, and the plot is shown in Fig. 5.5.

The theoretical value for the maximum deflection provided by reference[1], for a length-to-width ratio equal to  $3/2$  is :  $w = 0.1053 \text{ mm}$

Table 5.3: Center deflection of a uniformly loaded clamped plate.

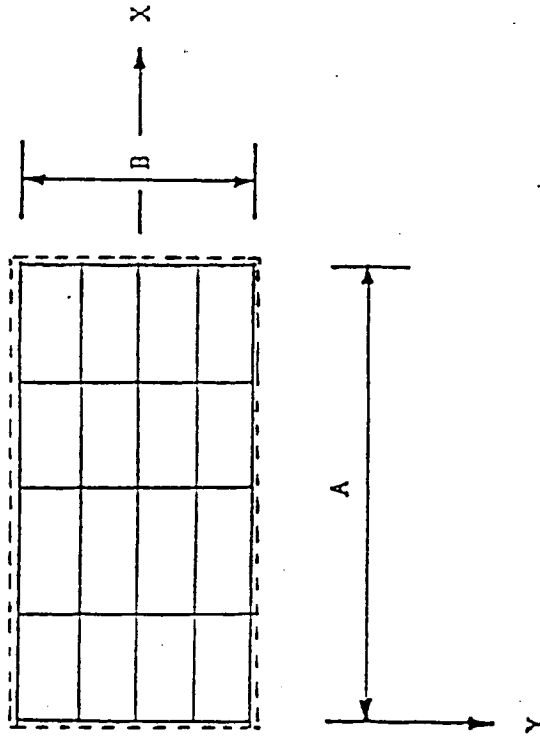
Number of elements	center deflection (in.)
4	0.1390
8	0.1113
16	0.1100
24	0.1080

### Center Deflection of a Uniformly Loaded Clamped Plate

Two dimensional bending finite elements based on the boolean sum are utilized for solution of a rectangular plate under uniformly distributed loads. The results obtained are in excellent agreement with the theoretical values even with a coarse mesh size.

# TEST CASE NUMBER 5

This test case consists of a rectangular pin - pin plate that has no rotation speed. This case will yield the free vibrations of the plate with no external loads being present. The mesh size is four elements square. Table 1 shows the numerical data obtained from D.A.R.T. and theoretical results.



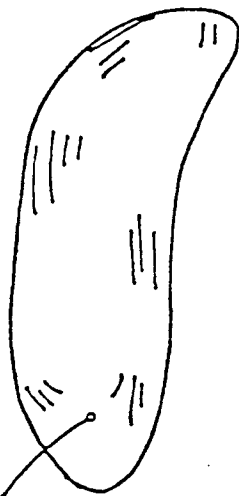
Setup Test 1  
4x4 case  
16 B.C.  
A= 10.0 in.  
B= 10.0 in.  
Thk=.2879 in.  
Angle= 90  
Speed= 0 rps  
Radius= 0 in.

MODE	Frequency (cps)		
	D.A.R.T. RESULTS	EXACT SOLN.	% ERROR
1	59.673	59.758	-.142
2	148.910	149.394	-.324
3	148.910	149.394	-.324
4	234.190	239.031	-2.03
5	299.050	298.788	+ .088

#### Natural Frequencies of a Square Plate

A simply-supported plate is analyzed for its natural frequency response. A mesh size of 4X4 was utilized for this purpose. The results obtained show very good agreement with the theoretical values.

$$P({}^{T+DT}\tilde{x}_1, {}^{T+DT}x, {}^{T+DT}x, {}^{T+DT})$$



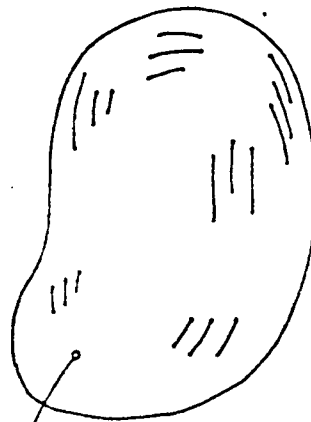
CONFIGURATION C  
( $T=T+DT$ )

$$P({}^T x_1, {}^T x_2, {}^T x_3, T)$$

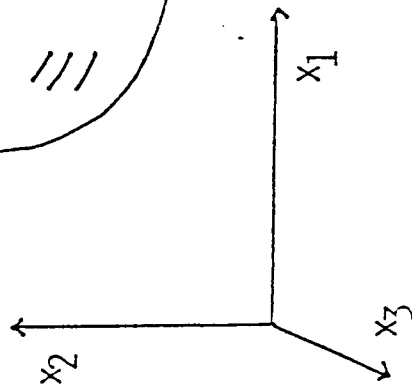


CONFIGURATION B  
( $T=T$ )

$$P({}^0 x_1, {}^0 x_2, {}^0 x_3, 0)$$



CONFIGURATION A  
( $T=0$ )

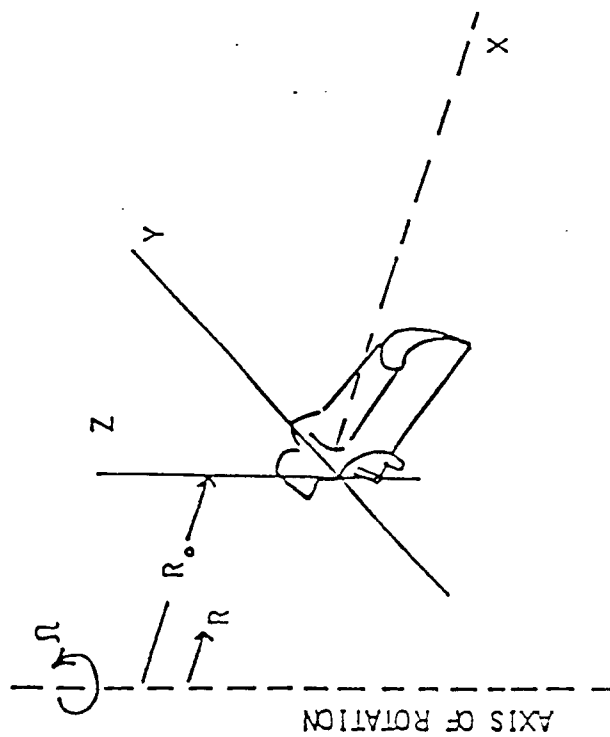


$$[{}^T K_L + {}^T K_{NL}]_{DU}^I = {}^{T+DT} R - \frac{{}^{T+DT} I - I}{{}^{T+DT} F}$$

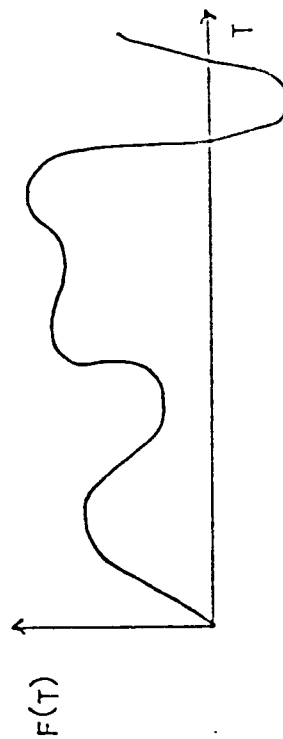
DESCRIPTION OF A BODY IN MOTION

#### DESCRIPTION OF A BODY IN MOTION

An Updated Lagrangian formulation is being utilized to account for large displacements and large rotations.



$$[M]\ddot{u} + [\dot{M}(\tau) + c]\dot{u} + [K]u = \{F(\tau)\}$$



## CENTRIFUGAL FORCES

$$\{F_C\} = \int_V \Omega \Omega^T \{F\} \, DV$$

$$\{F\} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \begin{Bmatrix} \rho \Omega_X^2 R_X \\ \rho \Omega_Y^2 R_Y \\ \rho \Omega_Z^2 R_Z \end{Bmatrix}$$

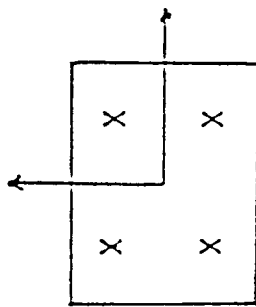
ELEVATED TEMPERATURES  
ANISOTROPIC MATERIAL



#### LOADING

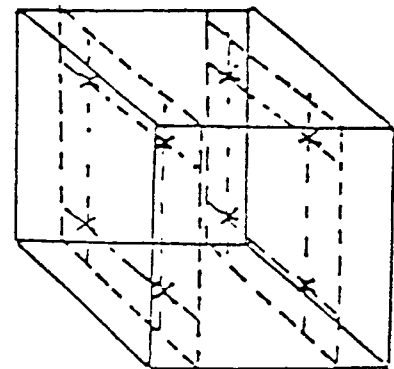
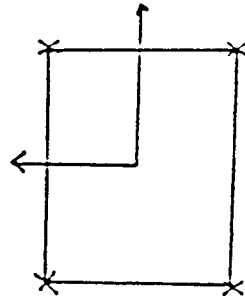
Elevated temperature effects, centrifugal forces and time dependent mass effects will be incorporated in the solution.

# NUMERICAL INTEGRATION

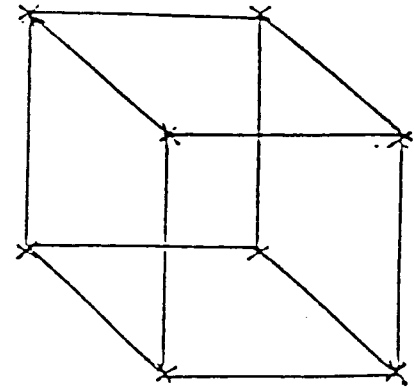


2-D

# HIDDEN-SYMBOLIC INTEGRATION



3-D



#### NUMERICAL INTEGRATION - HIDDEN SYMBOLIC INTEGRATION

The integrations involved in the element matrices are obtained exactly by utilizing a hidden-symbolic integration scheme that is developed here. As it is seen in the figure, the accuracy of the results will be improved since one can calculate the required nodal values at the nodes instead of at the numerical integration points.

ELEMENT

$$\begin{aligned}
 [K] &= \int \int \int \\
 [M] &= \frac{\partial}{\partial R} \\
 \{F\} &= \frac{\partial^2}{\partial R^2} \\
 &\vdots
 \end{aligned}$$

$\vec{L}_{\text{WYMONY}}^{\text{P}}$

HIDDEN-SYMBOLIC COMPUTATIONS	
POLDIF (P, IX, PX, NTERM, NDIMC)	
POLMLT ( . . . )	
POLADD ( . . . )	
POLINT ( . . . )	
POLIEV ( . . . )	
..	
.	
..	

EXACT

[K]  
[M]  
{F}  
...

BENEFITS

TIME

ACCURACY

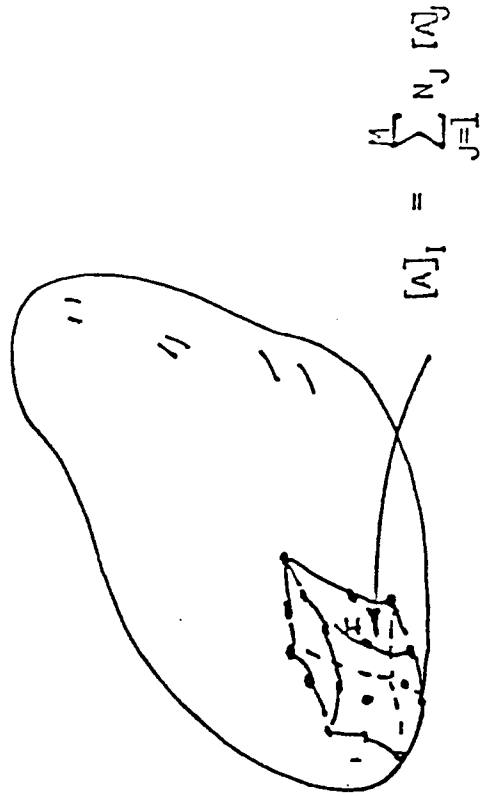
(P, II, "E" VERSION F.E.)

#### HIDDEN-SYMBOLIC COMPUTATIONS

A computer module is developed to operate on polynomials. These operations include differentiation, addition, multiplication, integration and integral evaluation. These modules are being utilized to formulate the finite element stiffness and mass coefficients exactly, "mp-version finite elements."

# MATERIAL CONSIDERATIONS

- MATERIAL PROPERTIES VARY AT ELEMENT NODAL POINTS
- $P(T, \sigma, x)$



$$\frac{P_M}{P_{M_0}} = \left[ \frac{T_M - T}{T_M - T_0} \right]^N \left[ \frac{s_F - \sigma}{s_F - \sigma_0} \right]^M \left[ \frac{\dot{s}_F - \dot{\sigma}}{\dot{s}_F - \dot{\sigma}_0} \right]^L$$

C.C.CHWIS, (1985)

#### MATERIAL CONSIDERATIONS

Unlike the traditional formulations, the material properties vary at the nodal points of the elements. The material properties are considered as functions of time, temperature, stress and the coordinates. The interpolation functions are for the geometry, for the material, and for the field variables. It may be noted that different interpolation functions are selected for interpolating the geometry, material and the field variables.

## CONCLUDING REMARKS

- NOVEL METHODS ARE UNDER INVESTIGATION FOR SOLUTION OF SINGULARITY PROBLEMS ENCOUNTERED IN MECHANICS. AT PRESENT, ONLY THE STRUCTURAL APPLICATIONS ARE BEING CONSIDERED. RELIABILITY ON THE RESULTS (THE STRUCTURAL RESPONSE) IS THE PRIMARY CONCERN. EFFICIENCY AND CAPACITY OF THE METHODS ARE BEING INVESTIGATED.
- MATERIAL PROPERTIES ARE MODIFIED TO INCLUDE  $T$ ,  $X$ ,  $\sigma$  EFFECTS. MATERIAL PROPERTIES VARY AT NODAL POINTS OF THE FINITE ELEMENTS FORMULATED. DIFFERENT CLASS OF INTERPOLATION AND BLENDING FUNCTIONS ARE UTILIZED IN ORDER TO REPRESENT MATERIAL BEHAVIOR UNDER ELEVATED TEMPERATURES WITHIN THE ELEMENT.
- TWO TRANSITION FINITE ELEMENTS ARE FORMULATED AND ARE BEING TESTED.
- A HIDDEN-SYMBOLIC COMPUTATION SCHEME IS DEVELOPED. FINITE ELEMENTS ARE BEING FORMULATED BY UTILIZING THIS SCHEME. INTEGRATIONS ARE PERFORMED EXACTLY IMPROVING ACCURACY AND EFFICIENCY.
- THE METHODS ARE ALSO APPLICABLE TO FLUIDS AND COUPLED PROBLEMS.

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